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# Assessment of Probability of Failure of Building Structures in Uncertainty Conditions

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**Abstract.** When assessing a reliability of building structures, probabilistic models are used that have some random values (RV) as their input. The statistical analysis of these RV inevitably introduces some uncertainties in the assessments of their parameters. In this paper a novel method is given to assess the probability of failure (PoF) of building structures (BS) in the conditions of epistemic uncertainty of the initial data. Epistemic uncertainty is a systemic uncertainty that emerges due to *insufficient* knowledge about the processes being studied and substitution of them by some approximation models. The uncertainty is accounted for using the Bayesian analysis.

## 1. Introduction

Design of building structures (BS) is conducted for the purpose of getting a guarantee that during their life cycle no limit state (failure) of any kind will occur. In the presented paper the limit state is related to the strength of the structure, which means that during the whole life cycle of the structure its bearing capacity has to be always larger than the loads and influences it is experiencing. Unfortunately, there always exist multiple *undeletable sources of uncertainties* in the loading as well as in the structure material properties, manufacturing technology, construction and, specifically, during their operation, which is strongly connected to the human and socio-economic factors. Hence, *it is necessary to account for these uncertainties when assessing the structural reliability.*

Currently uncertainties are classified as aleatory or as epistemic. The aleatory uncertainty, a.k.a. statistical uncertainty, is connected to the random character of the processes being studied. Aleatory uncertainty is the quintessence of randomness and reflects the unknown random results, which differ each time the same experiment is conducted. With accumulating factual data the uncertainty of this kind may be to some extent lowered, and characterized by the variance of the general set of data.

The quantitative assessment of the aleatory uncertainty can be relatively simple. In order to conduct this assessment, statistical modeling methods, like the Monte Carlo method [1,2], are used.

The epistemic uncertainty, a.k.a. the systemic uncertainty is the result, in the first place, of the absence of sufficient knowledge about the processes and substitution of the needed knowledge by some approximate models. Taking into consideration that our knowledge about the natural processes will always be not full, the epistemic uncertainty is here to stay with us forever. In order to assess this type of uncertainty such methods as Bayesian analysis, fuzzy logic and the theory of proof (the Dempster–Shafer theory [3-5] which is a generalization of the Bayesian theory [6-10] of subjective probability), are used.

In many problems of analysis of real systems and objects the prior probabilistic information about their state (condition) can be updated, after getting additional information from experiments, or as the result of observations of the system condition that corroborate or refute the prior information.

Many statistical problems have one trait in common, independent from the methods of their solution: even before getting a set of specific data, several probabilistic models are considered to be potentially acceptable for the considered process. After getting the initial data, some knowledge is created about the acceptability of these models. One of the ways to *reconsider* the relative acceptability of the probabilistic methods is to use the Bayesian approach which has at its core the well known Bayes' theorem.

## 2. Assessment of the structural reliability using the "load-resistance" scheme

When assessing reliability of building structures (and their elements) using the "load-resistance" scheme, the limit state function (LSF) has the form [11-15]:

$$G(R,S) = R - S$$

where $S$ are the generalized forces (or stresses) in the structure expressed via the external loads; $R$ is the generalized carrying capacity of the structure (its resistance).

The operational load from external sources $S$ is a RV. Acting on the system (or an element), it creates in the system some tensile, compression and/or transverse forces, bending and/or twisting moments, stresses, longitudinal or transverse overloads etc., as well as any of their combination [1]. Under bearing capacity we understand a force, bending or twisting moment, stress, pressure, overload, and deformation etc., that characterize the limit state of the element, which constrains its further usage [1].

Due to the fact that the load and the resistance are RVs (in general, random functions), the LSF is a function of RVs.

The condition of strength of building structures (BS) consists in that the LSF $G(R,S) > 0$ or $R > S$. The PoF $P_f$ of building structures (probability that the LSF ≤ 0) for the case when the load and the resistance both have a normal PDF, is calculated using the formula [11-15]

$$P_f = \Phi(-\beta) \tag{1}$$

where $\Phi(x)$ is the standard normal probability density function (PDF) and $\beta$ is the reliability index.

Let $\mu_G, \sigma_G$ be the mathematical expectation and the standard deviation of the RV $G(R,S)$ correspondingly. Then the reliability index $\beta$ is calculated using formula

$$\beta = \frac{\mu_G}{\sigma_G} \tag{2}$$

In the general case the LSF can be a function of many variables. Consider a LSF as a continuous and differentiable function of many random arguments $G(x_1,..,x_n)$. Using the method for linearization of a function of multiple variables [16], which permits calculating the numerical characteristics of the LSF using the numerical characteristics of its arguments, the assessments of the mathematical expectation and variance of the LSF are calculated from formulas:

$$\mu_G = G\left(m_{x_1}, m_{x_2}, .., m_{x_n}\right)$$

$$\sigma_G^2 = \sum_{i=1}^{n} \left[ \left( \frac{\partial G}{\partial x_i} \right)^2_{m_{x_1},..,m_{x_n}} \cdot \sigma_{x_i}^2 \right]$$

where $m_{x_i}, \sigma_{x_i}^2$ are the mathematical expectation and variance of the LSF arguments accordingly.

Knowing the mathematical expectation and the variance of LSF it is possible to calculate, using formula (2) , the reliability index, and further, via formula (1), to assess the PoF of the BS.

## 3. Description of the aleatory and epistemic uncertainties when assessing the reliability using the “load-resistance” scheme.

Consider $X = (X_1,..,X_n)$ as a vector of the LSF input parameters, $f_G(g)$ as the PDF of the LSF.

Each RV $X_i$ is described by its own set $\theta$ of statistical parameters. If all the RVs $X_i$ contain aleatory uncertainty, i.e., the statistical parameters $\theta$ are fully known (either through a large set of data or from already acquired knowledge about the initial parameters), then the problem is formulated as an ordinary reliability assessment, i.e., defining the deterministic value of the probability of the event $G(X) \le 0$ . The graphical interpretation of this process is given in Figure 1 [17].

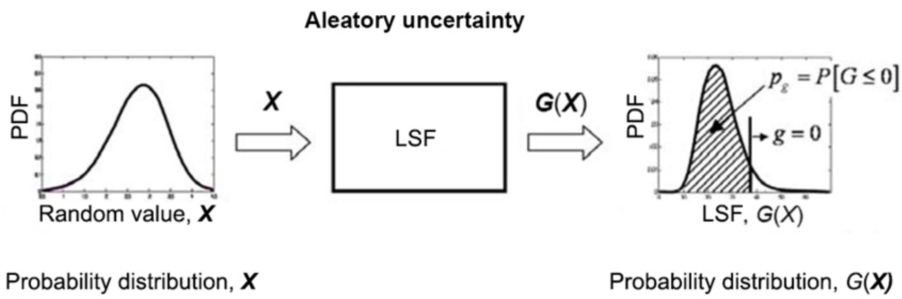


**Figure 1**. Aleatory uncertainty when assessing reliability of building structures.

If a part or all of the input variables X contain the epistemic uncertainty due to lack of needed data or knowledge, then the corresponding statistical parameters become uncertain, which leads to uncertainty in the assessment of reliability. In this case the parameters are considered random. The graphic interpretation of this process is shown in Figure 2 [17].

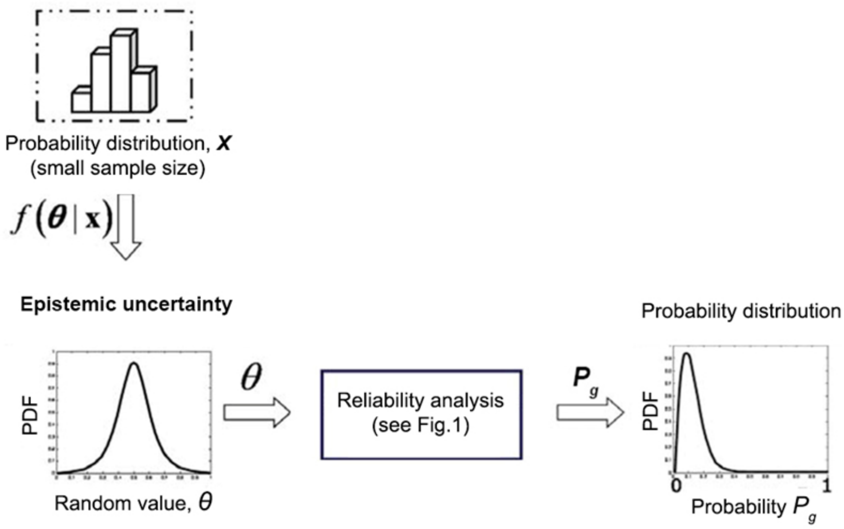


**Figure 2**. Epistemic uncertainty when assessing reliability

In this case the RVs of the vector X are substituted by corresponding parameters , and, as the result of the analysis, we obtain the LSF G( ) and the probability . Due to the fact that the statistical parameters are random, the probability also will be random and have a PDF. In this case for each set of statistical parameters there will be a different assessment of reliability. At the end, using the distribution of parameters , we obtain the PDF of .

## 4. Core elements of the Bayesian approach

The basic of the Bayesian approach in statistics is the Bayes' theorem. Consider observation of a RV *Y* that has $p(y|\theta)$ as its PDF with parameters $\theta$ and the goal is to define the RV $\theta$, which has an unknown PDF $p(\theta)$. As a result of some observations a set of statistical data $y$ is obtained (values of the RV *Y*). The conditional probability is calculated from Bayes' formula

$$p(\theta|y)=\frac{p(y|\theta)p(\theta)}{p(y)}$$

The distribution $p(\theta)$ is called the prior distribution of probabilities of possible values of parameter $\theta$ (this PDF is adopted *before* the statistical data is available). The distribution $p(y|\theta)$ is called posterior PDF for $\theta$ on the condition that data *y* was observed (this PDF is calculated *after* the statistical data becomes available).

Hence, the Bayesian approach uses as the initial data two types of information: 1) prior information, which is contained in the initial (sample) statistical data. At that the prior information is presented as some prior PDF of the analyzed unknown parameter, which describes the level of certainty in that this parameter will take this or that value, *before* using the statistical data; 2) new posterior data. With the arrival of this data initial PDF is specified (recalculated) using the Bayes' formula, and the prior distribution becomes posterior distribution.

When using the Bayes' approach apart from using the PDF of the considered RV Y the prior PDF of parameters $\theta$ of the PDF of RV *Y* are also utilized. Relying on the statistical data, the prior PDF of parameters $\theta$ is modified by multiplying it by the likelihood function and then normalizing the product. The result is the posterior distribution of the parameters $\theta$. In other words, the distribution parameters are themselves RVs with some PDFs; hence, here the uncertainty is of the second order (epistemic): «random parameters of a RV» or «distribution of the parameters of the distribution».

Usually the prior distributions are classified as informative, not informative and connected. The informative prior distribution is expressing specific information about a RV *before* using the initial statistical data. Prior distribution often is assigned subjectively.

The not informative prior distribution or prior objective distribution expresses fuzzy or general information about a RV, i.e., the properties of the PDF are not set subjectively. *In other words, this is the case of total absence of prior information*.

## 5. Application of the Bayesian approach when assessing the reliability using the "load-resistance" scheme

Assume that the generalized structure resistance *R* and load *S* are random and are distributed normally. Let the mathematical expectations $\mu_R, \mu_S$ of these quantities be unknown, but let the variances $\sigma_R^2, \sigma_S^2$ be known, i.e., $\mu_R, \mu_S$ are considered as independent RVs. Assume that they have *not informative prior distribution*.

According to the Bayes' approach [18, 19] the posterior distribution of $\mu_R, \mu_S$ will be a normal PDF with parameters $N\left(\overline{R}, \frac{\sigma_R}{\sqrt{n}}\right)$, $N\left(\overline{S}, \frac{\sigma_S}{\sqrt{n}}\right)$ correspondingly, where *n* is the sample size $\overline{R}, \overline{S}$ are the sample averages of resistance and load. Because $\mu_R, \mu_S$ are normally distributed the quantity

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$$\beta = \frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}$$

will also be distributed normally (as the difference of two normally distributed quantities) with following parameters (mathematical expectation, variance and standard deviation):

$$\mu_\beta = \frac{\bar{R} - \bar{S}}{\sqrt{\sigma_R^2 + \sigma_S^2}};$$

$$\sigma_\beta^2 = \frac{1}{\sigma_R^2 + \sigma_S^2}\left(\frac{\sigma_R^2}{n} + \frac{\sigma_R^2}{n}\right) = \frac{1}{n}; \quad \sigma_\beta = \frac{1}{\sqrt{n}}.$$

The PDF of RV $\beta$ is

$$f_\beta(x) = \frac{\sqrt{n}}{\sqrt{2\pi}} \exp\left(-\frac{n}{2}\left[x - \frac{\bar{R} - \bar{S}}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right]^2\right) \tag{3}$$

Because $P_f = \Phi(-\beta) = y(\beta)$ and function $\Phi(x)$ is a non decreasing monotonous function, then, according to the theorem [20] about the PDF of a function of a RV, the PDF of failure $P_f$ has the form:

$$f_{P_f}(p) = \left(y^{-1}(p)\right)' f_\beta\left(y^{-1}(p)\right) \tag{4}$$

The function, inverse to function *y,* will be

$$y^{-1}(p) = \Phi^{-1}(1-p)$$

Out of formulas (3), (4) we get that the PDF for the probability of failure has the form

$$f_{P_f}(p) = \sqrt{n} \exp\left(-\frac{n}{2}\left[\Phi^{-1}(1-p) - \frac{\bar{R} - \bar{S}}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right]^2 + \frac{\left[\Phi^{-1}(1-p)\right]^2}{2}\right)$$

Then the mathematical expectation and the variance of the probability of failure will be

$$M\left[P_f\right] = \int_0^1 p f_{P_f}(p)\, dp,$$

$$D^2\left[P_f\right] = \int_0^1 f_{P_f}(p)\left(p - M\left[P_f\right]\right)^2 dp.$$

Since the probability of failure is connected with reliability by relation $P_S = 1 - P_f = \Phi(\beta)$, the PDF of reliability will be

$$f_{P_S}(p) = -\sqrt{n} e\,\mathrm{xp}\left(-\frac{n}{2}\left[\Phi^{-1}(p) - \frac{\bar{R} - \bar{S}}{\sqrt{\sigma_R^2 + \sigma_S^2}}\right]^2 + \frac{\left[\Phi^{-1}(p)\right]^2}{2}\right)$$

In this expression the quantity $\tilde{\beta} = \frac{\overline{R} - \overline{S}}{\sqrt{\sigma_R^2 + \sigma_S^2}}$ can be considered as the sample (conditional) reliability index.

Let it be necessary, that the system PoF would be no more than an allowable value $p_l$ (for instance, $10^{-6}$) with given (prescribed) probability $p^*$ (confidence level). Then, from equation

$$P\left(P_f < p_l\right) = \int_0^{p_l} f_{P_f}\left(p\right) dp = p^*$$

it is possible to find the conditional reliability index $\tilde{\beta}$ depending on the sample size $n$:

$$\sqrt{n}\int_0^{p_l} \exp\left(-\frac{n}{2}\left[\Phi^{-1}\left(1-p\right) - \tilde{\beta}\right]^2 + \frac{\left[\Phi^{-1}\left(1-p\right)\right]^2}{2}\right) dp = p^* \qquad (5)$$

Hence, we obtained the conditional reliability index, which provides the prescribed reliability level with given confidence level depending on the sample size.

Formula (5) reflects the influence of the epistemic uncertainty on the PoF reliability of the system. In Fig. 3 the dependence of the conditional safety characteristic $\tilde{\beta}$ is given as a function of the sample size, which provides the probability of failure not larger than $p_l = 10^{-6}$ with probability $p^* = 0.99$.

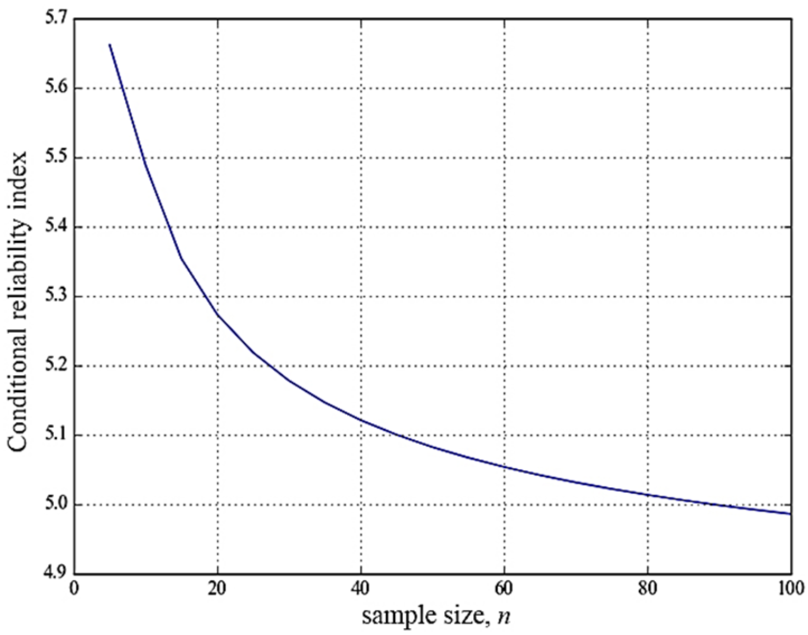


**Figure 3**. Dependence of the conditional reliability index on the sample size which provides the PoF value not larger than $10^{-6}$ with probability 0.99

It should be noted that in the classical case (which does not account for the uncertainties of the distribution parameters) the value of $\beta$ that corresponds to the PoF= $10^{-6}$ is equal to 4.75.

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